JOURNAL OF DIFFERENTIAL EQUATIONS 21, 1–8 (1976)

On the Existence and Uniqueness of Periodic Motions*

JORGE LEWOWICZ†

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Buenos Aires, Argentina

Received August 12, 1974

DEDICATED TO PROFESSOR JULIO RICALDONI

1

The problem of finding analytic conditions for the existence and uniqueness of periodic motions of an autonomous differential equation is considered in [5, p. 75], where some results of V. Korolov concerning the existence of an attracting periodic orbit are stated. As it is easy to show, the closed orbits encountered by Korolov [1, 2] are *elementary* [6] attracting periodic motions.

In this paper we give necessary and sufficient conditions for the existence of an elementary attracting periodic motion of an autonomous system as the only invariant set contained in a given positively invariant region Ω of phase space.

In a previous work (see [3, section 5]) we have obtained other results in the same direction; however, here our purpose is to find conditions in terms of the second member of the differential equation and in such a way that the open character of the property under consideration can be recognized at once by mere inspection of each one of them. As a consequence, our present results look more satisfactory, at least from the "quantitative" point of view, than the results of [3] mentioned above (see Remarks (iii)–(v) below).

On the other hand, the results of Korolov refer to a situation less general than the one considered here: his Ω is the product of an annulus in $R^2 = \{(x, y)\}$ by an $(n - 2)$-cell and the 1-form $-y\,dx + x\,dy$ is assumed to be monotone on the solutions of the differential equation. Furthermore, he only obtains sufficient conditions for the existence of an attracting periodic motion (see also Remark (v) at the end of the paper).

* This research was supported partially by the Centro Regional de Matemática para América Latina, and by the Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Buenos Aires, Argentina. To both institutions I express my gratitude.

† Present address: Instituto de Matemática e Estatística, Cidade Universitária "Armando de Salles Oliveira," Caixa Postal 20.570-Agencia Iguatemí, São Paulo, Brasil.

Let us say, finally, that although our results concern analytic vector fields defined in R^n, they can be generalized, using the same methods, for C^2-flows taking place in a differentiable manifold.

2

Let X be a real analytic vector field defined in some open subset of R^n and $\varphi(x, t)$, $t \in R$, the solution of $\dot{x} = X(x)$ through x, i.e., $\varphi(x, 0) = x$. Since X is analytic, it may be extended to some open subset of C^n; we shall sometimes consider the real solution $\varphi(x, t)$ through a point $x \in C^n$ and also the complex solution $\varphi(x, t + si)$, t, $s \in R$, through such a point. Also, we shall still write C^n for the $2n$-dimensional differentiable manifold determined by (C^n, h), where h: $C^n \to R^{2n}$ is defined by

$$h(x_1, ..., x_n) = (\tfrac{1}{2}(x_1 + \bar{x}_1), -\tfrac{1}{2}i(x_1 - \bar{x}_1), ..., \tfrac{1}{2}(x_n + \bar{x}_n), -\tfrac{1}{2}i(x_n - \bar{x}_n)).$$

A function f defined in some subset of C^n and with values in C^m will be called real if $f(x) \in R^m$ for $x \in R^n$. Vector fields and differential forms will be called real if their coordinates with respect to the corresponding standard basis are real functions. Manifolds or submanifolds will be considered without boundary unless otherwise stated. (The rest of the terminology and notation is also standard.)

Let Ω be an n-dimensional differentiable compact submanifold with boundary of R^n contained in the domain of definition of X. For brevity, we shall say that Ω is admissible if at each x of the boundary of Ω, the vector field X points to the interior of Ω. If Ω is admissible, it contains a maximal φ-invariant set, say I, which is compact.

Now we shall state some of the results of [3] that will be applied afterward. An attractor like I will be called C-generic if it is stable as an invariant subset of the real extension of the flow to a C^n-neighborhood of Ω; see [3, p. 183].

(a) *Assume that I is a periodic motion. Then I is elementary if and only if it is C-generic.*

Proof. See [3, Theorem 1].

(b) *Assume that I is C-generic. Then there exists a C^n-neighborhood E of Ω and a sequence $t_n \to \infty$ such that $\varphi(x, t_n)$ converges uniformly on E to a real analytic retraction α. Furthermore, $\alpha(\Omega) = I$ and I is an analytic submanifold of R^n.*

Proof. See [3, Lemmas 1, 2, and 3].

(c) *If I is C-generic, it is a deformation retract of Ω.*

Proof. It follows from Lemma 4 of [3].

3

Let w be a real analytic closed 1-form defined in a neighborhood of Ω and such that

$$w_x(X) > 0$$

for each $x \in \Omega$. If v is a real potential function of w, then v is defined, in a suitable C^n-neighborhood of Ω, modulo real numbers, and consequently $\mathrm{Im}(v)$ is there a well-defined function. Since

$$d(\mathrm{Im}(v)) = \mathrm{Im}(w),$$

we have that

$$d(\mathrm{Im}(v))_x(iX) = \mathrm{Re}(w_x(X)),$$

which is different from zero if $x \in C^n$ is close enough to Ω. By the implicit function theorem it follows that there is an open C^n-neighborhood of Ω whose intersection with $\{x \in C^n\colon \mathrm{Im}(v(x)) = 0\}$ is a connected $(2n-1)$-dimensional submanifold M of C^n. To each Ω and w in the above conditions we associate in this way, a submanifold $M(\Omega, w)$ of C^n contained in the domain of X and w, and such that $w_x(X) \neq 0$ for $x \in M(\Omega, w)$.

If $w_x(X) = 1$ for $x \in \Omega$, $w(X) = 1$ in a C^n-neighborhood of $M(\Omega, w)$ and in this case the vector field X is tangent to M. In fact, from

$$v(\varphi(x, t)) - v(x) = \int_0^t w_{\varphi(x,u)}(X)\, du = t$$

it follows that

$$\mathrm{Im}(v(\varphi(x, t))) = \mathrm{Im}(v(x));$$

thus, if $x \in M(\Omega, w)$ and t is small enough, $\varphi(x, t)$ also belongs to $M(\Omega, w)$.

Proposition 1. *Assume that Ω is admissible and that $w_x(X) = 1$ for $x \in \Omega$. Then I is an elementary attracting periodic motion if and only if the following conditions are fulfilled:*

(1) *The integral homology groups of Ω are those of S^1.*

(2) *There exists a compact differentiable $(2n-1)$-dimensional submanifold with boundary H of $M(\Omega, w)$, $\Omega \subset H$, such that for each x in the boundary of H, $X(x)$ points to the interior of H.*

Proof. First we show that the existence of such an H is necessary. By (b), the assumptions imply the existence of a real analytic retraction α that commutes with the flow. Inasmuch as the range of α is a 1-dimensional complex analytic manifold, if $x \in I$ and t, s are real numbers of small absolute value, then $\varphi(x, t + is)$ belongs to the range of α since

$$\varphi(x, t + is) = \varphi(\alpha(x), t + is) = \alpha(\varphi(x, t + is)).$$

Because X does not vanish in Ω, it follows that if $x \in E$ and is close enough to Ω, there exists $y \in I$ and $s \in R$, such that

$$\alpha(x) = \varphi(y, is).$$

In particular, if $x \in M(\Omega, w)$ we must have $s = 0$, since

$$\begin{aligned} 0 &= \lim \operatorname{Im}(v(\varphi(x, t_n))) = \operatorname{Im}(v(\alpha(x))) \\ &= \operatorname{Im}\left(\int_0^s w_{\varphi(y,iu)}(iX)\, du\right) = s. \end{aligned}$$

Therefore there exists a compact M-neighborhood N of Ω with the properties that $\alpha(N) = I$ and that the real solution through each point of N is Lyapunov stable (see [3, p. 185]); consequently

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(x, t), I) = 0$$

for each $x \in N$. Moreover, on account of the elementary character of the periodic orbit I, it is easy to prove that $\| \varphi(x, t) - \varphi(\alpha(x), t)\|$ converges exponentially and uniformly to zero for $x \in N$. Then we may show as, in [4, Theorem 8], that

$$V(x) = \int_0^\infty \| \varphi(x, t) - \alpha(\varphi(x, t))\|^2\, dt$$

is differentiable and that $dV_x(X)$ is negative for $x \in N$, $x \notin I$.

Consider now the restriction of X to N, and let $f\colon N \to R$ be a C^∞ function that vanishes on the boundary of N and is positive elsewhere. The vector field fX defines a new flow $\psi(x, t)$ on N and if x belongs to the interior of N, the same happens to $\psi(x, t)$ for each $t \leqslant 0$. Now choose $\epsilon > 0$ such that $V_\epsilon = \{x \in N\colon V(x) \leqslant \epsilon\}$ is a compact set contained in the interior of N and let $T < 0$ be such that $\psi(V_\epsilon, T) \supset \Omega$, which is possible since $\varphi(\Omega, t) \subset V_\epsilon$ if t is large enough. Then $H = \psi(V_\epsilon, T)$ is the required manifold.

Since the necessity of condition (1) is an immediate consequence of (c) we prove next the "if" part.

Let $\sigma > 0$ be such that $\varphi(x, is)$ is defined and uniformly bounded for

$x \in H$ and $|s| < \sigma$. The function $g: H \times (-\sigma, \sigma) \to C^n$, $g(x, s) = \varphi(x, is)$ is locally a homeomorphism since from

$$\mathrm{Im}(v(\varphi(x, is))) = s + \mathrm{Im}(v(x))$$

it follows readily that $\varphi(x, is) \in M$ if $x \in M$ and $s \neq 0$.

Therefore, the range of g contains a C^n-neighborhood of Ω and as $\varphi(\varphi(x, is), t) = \varphi(\varphi(x, t), is)$ and (2) implies that $\varphi(x, t) \in H$ for $x \in H$, $t \geqslant 0$, we may deduce that the real solutions starting from that C^n-neighborhood of Ω remain uniformly bounded in the future. Thus, I is a C-generic attractor and hence, by (b), an analyticial submanifold of R^n. Now condition (1) and (a) imply that I is an elementary attracting periodic motion.

THEOREM 1. *Let X be an analytic vector field and Ω an admissible subset of R^n. Then the following conditions are necessary and sufficient for I to be an elementary attracting periodic motion*:

(1) *The integral homology groups of Ω are those of S^1.*

(2) *There exists a real analytic closed 1-form w, such that $w_x(X) > 0$ for $x \in \Omega$.*

(3) *There exists a compact differentiable $(2n - 1)$-dimensional submanifold with boundary H of $M(\Omega, w)$, $\Omega \subset H$, such that for each x in the boundary of H, the vector field $(w_x(X))^{-1}X$ points to the interior of H.*

Proof. Necessity. I being a periodic orbit, we define on it the closed 1-form θ, by $\theta_x(X) = 1$ for each $x \in I$. If α is the analytic retraction commuting with the flow, given by (b), $w = \alpha^*(\theta)$ satisfies (2) since

$$w_x(X) = \theta_{\alpha(x)}(\alpha_x'(X)) = \theta_{\alpha(x)}(X) = 1.$$

On account of the previous proposition this proves the necessity.

Sufficiency. Proposition 1 may be applied to

$$Y = (w_x(X))^{-1}X$$

since $w(Y) = 1$. The sufficiency follows from the fact that the real vector fields X and Y are geometrically equivalent when restricted to R^n.

COROLLARY 1. *Let X be an analytic vector field defined in R^n. Then X has an elementary attracting periodic motion if and only if there exists an admissible Ω and a differential form w satisfying conditions* (1)–(3) *of Theorem* 1.

Proof. This result is an easy consequence of Theorem 1 and of the existence of monotone Lyapunov functions for an attracting periodic motion.

Remarks. (i) It is clear that if $X^{\#}$ and $w^{\#}$ are C^0-close enough to X and w in a suitable C^n-neighborhood of Ω, then conditions (1), (2), or (3) of Theorem 1 will be fulfilled by $X^{\#}$ and $w^{\#}$, provided they are satisfied by X and w.

(ii) From the proof of (c), the integral homolgy groups of H are also those of S^1. On the other hand, for sufficiency, it is enough that the solutions starting from H remain uniformly bounded in the future, since this implies that a certain C^n-neighborhood of H will have the same property.

(iii) Let $X = (P(u, v), Q(u, v))$ be a real analytic vector field in R^2 such that $d\theta_{(u,v)}(X) > 0$ for $(u, v) \neq (0, 0)$; here $\theta = tg^{-1}(v/u)$. Then, the manifold H is contained in $M = \{(u, v) \in C^2 \colon u\bar{v} = \bar{u}v\}$. The mapping $k\colon R^3 \to C^2$,

$$k(x, y, z) = (x(1 + iz), y(1 + iz))$$

is injective in the complement of the z-axis and every $(u, v) \in M$ is in the range of k, except for those $(u, v) \neq (0, 0)$ with $\mathrm{Re}(u) = \mathrm{Re}(v) = 0$.

If $Y = (S(u, v), T(u, v))$ is the vector field

$$Y = (d\theta_{(u,v)}(X))^{-1}X,$$

then $(k^{-1})'(Y)$ is a vector field Z defined in some subset of R^3 and invariant under the symmetry $(x, y, z) \to (x, y, -z)$. The differential equation defined by Z is

$$\begin{aligned}
\dot{x} &= \tfrac{1}{2}(S(x(1 + iz), y(1 + iz)) + S(x(1 - iz), y(1 - iz))),\\
\dot{y} &= \tfrac{1}{2}(T(x(1 + iz), y(1 + iz)) + T(x(1 - iz), y(1 - iz))),\\
\dot{z} &= (2ix)^{-1}((1 - iz)\, S(x(1 + iz), y(1 + iz))\\
&\qquad - (1 + iz)\, S(x(1 - iz), y(1 - iz)))\\
&= (2iy)^{-1}((1 - iz)\, T(x(1 + iz), y(1 + iz))\\
&\qquad - (1 + iz)\, T(x(1 - iz), y(1 - iz))),
\end{aligned}$$

and our problem consists in finding the Z positively invariant bounded regions of R^3 that are symmetric with respect to the plane $\{(x, y, z) \in R^3 \colon z = 0\}$. Moreover, instead of the positive invariance, it is enough to have that the solutions issuing from such a region can be defined in the future. In fact, since on M, $u \neq 0$ $(v \neq 0)$ and $u \in R$ $(v \in R)$ imply $v \in R$ (resp. $u \in R$), it is easy to show that there are values u_1, u_2 (resp. v_1, v_2) such that no solution starting from our region can reach either the plane $u = u_1$ $(v = v_1)$ or the plane $u = u_2$ $(v = v_2)$. Then we may show, as in [3, Proof of Corollary 2], that these solutions remain uniformly bounded in the future.

(iv) To look for a closed 1-form w satisfying $w_x(X) > 0$, for $x \in \Omega$, we may pick a real analytic 1-form, $w^{\#}$, closed but not exact, and try to find a real constant c and a real polynomial P, such that

$$w = cw^{\#} + dP$$

satisfies the required properties.

(v) We may get several results of the type of those obtained by Korolov, in the following way. Choose a closed 1-form w defined on a given Ω (with the homology of S^1) and consider an $M(\Omega, w)$. Next take a differentiable function $V: C^n \to R$ such that $\Omega \subset \{x \in C^n: V(x) \leqslant 0\}$ and that the intersection

$$H = \{x \in C^n: V(x) \leqslant 0\} \cap M(\Omega, w)$$

is compact. Then we may state: *If X is a real analytic vector field for which Ω is admissible, $w_x(X) \neq 0$ for $x \in H$, and $(w(X))^{-1} X(V) < 0$ for $x \in \{x \in C^n: V(x) = 0\} \cap M(\Omega, w)$, then the maximal invariant set contained in Ω is an elementary attracting periodic motion.* As an example, consider $R^3 = \{(x, y, z)\}$ and the 1-form $r^{-1}(-y\, dx + x\, dy)$, where $r = x^2 + y^2$. Let $M = \{(x, y, z) \in C^3: \operatorname{Im}(y\bar{x}) = 0\}$ and $V: C^3 \to R$, $V(x, y, z) = |\, x^2 + y^2 - 1\,|^2 + |\, z\,|^2 - k$, where k is a real number. Then

$$H = \{(x, y, z) \in C^3: V(x, y, z) \leqslant 0 \text{ and } \operatorname{Im}(y\bar{x}) = 0\}$$

is compact. If $X = (f, g, h)$ is a real analytic vector field, $-yf + xg \neq 0$ on H and

$$\operatorname{Re}(r(-yf + xg)^{-1} (2(xf + yg)\overline{(x^2 + y^2 - 1)} + \bar{z}h)) < 0$$

for $|\, x^2 + y^2 - 1\,|^2 + |\, z\,|^2 = k$, the assumptions of the above statement are satisfied. This is the case if, for instance, $f = -y - x(r - 1)$, $g = x - y(r - 1)$, $h = -2rz + (r - 1)$ and k is chosen conveniently.

REFERENCES

1. V. V. KOROLOV, On the uniqueness of a limit cycle for higher dimensional autonomous systems, *Dopovidi Akad. Nauk Ukrain. RSR* (1964), 1430–1433; *MR* **31** (1966), 431.
2. V. V. KOROLOV, The existence and uniqueness of a periodic solution of many dimensional autonomous systems, *Ukrain Mat. Z.* **17** (1965), 47–60; *MR* **33** (1967), 350.
3. J. LEWOWICZ, Stability properties of a class of attractors, *Trans. Amer. Math. Soc.* **185** (1973), 183–198.

4. J. L. Massera, On Liapounoff's conditions of stability, *Ann. of Math.* **50** (1949), 705–721; *MR* **11**, 721.
5. R. Reissig, G. Sansone, and R. Conti, "Nichtlineare Differentialgleichungen Hoherer Ordnung," Edizioni Cremonese, Roma, 1969.
6. S. Smale, Stable manifolds for differential equations and diffeomorphisms, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 97–116; *MR* **29**, 2818b.